

FIRST- AND SECOND-ORDER WAVE GENERATION THEORY

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Abstract. The first-order and the second-order wave generation theory is studied in this paper. The theory is based on the fully nonlinear water wave equations. The nonlinear boundary value problem (BVP) is solved using a series expansion method. Using this method, the problem becomes a set of linear, signalling problems according to the expansion order. The first-order theory leads to a homogeneous BVP. It is a system with the first-order steering of the wavemaker motion as input and the surface wave field with propagating and evanescent modes as output. The second-order theory leads to a nonhomogeneous BVP. It is a system where the second-order steering of the wavemaker motion is prescribed in such a way that the second-order part of the surface elevation far from the wavemaker contains only the bound wave component and the free wave component vanishes. The second-order surface wave elevation consists of a superposition of bichromatic frequencies.

1. INTRODUCTION

In this paper, we will consider the problem how to generate waves in a wave tank of a hydrodynamic laboratory. The wave tank in the context of this paper is a facility with a wavemaker on one side and an artificial wave absorbing beach on the other side. We consider a tank with a flat bottom, and no water is flowing in or out of the tank. Typically, the situation is that waves are generated by a flap type wavemaker at one side of a (long) tank; the motion of the flap ‘pushes’ the waves to start propagating along the tank. This means that typically, we are dealing with a signalling problem or a boundary value problem (BVP), which is different from an initial value problem (IVP) when one tries to find the evolution of waves from given surface elevation and velocities at an initial moment. To illustrate this for the simplest possible case, consider the linear, non-dispersive second-order wave equation for waves in one spatial direction x and time t : $\partial_t^2 \eta = c^2 \partial_x^2 \eta$. Here, η denotes the surface wave elevation, and $c > 0$ is the constant propagation speed. The general solution is given by $\eta(x, t) = f(x - ct) + g(x + ct)$, for arbitrary functions f and g . The term $f(x - ct)$ is the contribution of waves travelling to the right (in the positive x -direction), and $g(x + ct)$ waves running to the left. For the IVP, specifying at an initial time (say at $t = 0$) the wave elevation $\eta(x, 0)$ and the velocity $\partial_t \eta(x, 0)$ determines the functions f and g uniquely. For the BVP, resembling the generation

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at $x = 0$, we prescribe the wave elevation at $x = 0$ for all positive time, assuming the initial elevation to be zero for positive x (in the tank, this means a flat surface prior to the start of the generation). If the signal is given by $s(t)$, vanishing for $t < 0$, the corresponding solution running into the tank is $\eta(x, t) = f(x - ct)$ which should be equal to $s(t)$ at $x = 0$, leading to $\eta(x, t) = s(t - x/c)$. Conversely, for a desired wave field $f(x - ct)$ running in the tank, the required surface elevation at $x = 0$ is given by $s(t) = f(-ct)$. This shows the characteristic property of the BVP for the signalling problem.

The actual equations for the water motion and the precise incorporation of the flap motion are much more difficult than shown in the simple example above. In particular, for the free motion of waves, there are two nontrivial effects. The first effect is dispersion: the propagation speed of waves depends on their wavelength (or frequency), described explicitly in the linear theory (i.e. for small surface elevations) by the linear dispersion relation (LDR), see formula (5). In fact, for a given frequency, there is one normal mode that travels to the right as a harmonic wave, the propagating mode, and there are solutions decaying exponentially for increasing distance (the evanescent modes). We will use the right propagating mode and the evanescent modes as building blocks to describe the generation of waves by the wavemaker. For each frequency in the spectrum, the Fourier amplitude of the flap motion is then related to the amplitude of the corresponding propagating and evanescent modes. If we assume these amplitudes to be sufficiently small, say of the ‘first order’ ϵ , with ϵ is a small quantity, we will be able to deal with nonlinear effects in a sequential way. This is needed because the second effect is that in reality, the equations are nonlinear. The quadratic nature of the nonlinearity implies that each two wave components will generate other components with an amplitude that is proportional to the product of the two amplitudes, the so-called ‘bound wave components’ which have amplitudes of the order ϵ^2 . These are the so-called ‘second-order effects’. For instance, two harmonic waves of frequency ω_1, ω_2 and wavenumber k_1, k_2 (related by the LDR) will have a bound wave with frequency $\omega_1 + \omega_2$ and wavenumber $k_1 + k_2$. Since the LDR is a concave function of the wavenumber, this last frequency-wavenumber combination does not satisfy the LDR, i.e. this is not a free wave: it can only exist in the combination of the free wave. This second-order bound wave that comes with a first-order free wave has also its consequence for the wave generation. If the first-order free-wave component is compatible with the flap motion, the presence of the bound wave component will disturb the wave motion, such that the additional second-order free-wave will be generated as well. This is undesired, since the second-order free wave component has a different propagation speed as the bound wave component, thereby introducing a spatially inhomogeneous wave field. That is why we add to the flap motion the additional effects of second-order bound waves, thereby preventing any second-order free-wave component to be generated. This process is called ‘*second-order steering*’ of the wavemaker motion.

This technique can be illustrated using a simple IVP for an ordinary differential equation as follows. Consider the nonlinear equation with a linear operator \mathcal{L}

$$\mathcal{L}\eta := \partial_t^2 \eta + \omega_0^2 \eta = \eta^2,$$

for which we look for small solutions, say of order ϵ , a small quantity. The series expansion technique then looks for a solution in the form

$$\eta = \epsilon \eta^{(1)} + \epsilon^2 \eta^{(2)} + \mathcal{O}(\epsilon^3).$$

Substitution in the equation and requiring each order of ϵ to vanish leads to a sequence of IVPs, the first two of which read:

$$\mathcal{L}\eta^{(1)} = 0; \quad \mathcal{L}\eta^{(2)} = (\eta^{(1)})^2; \quad \dots$$

Observe that the equations for $\eta^{(1)}$ and $\eta^{(2)}$ are linear equations, homogeneous for $\eta^{(1)}$ and nonhomogeneous (with known right-hand side after $\eta^{(1)}$ has been found) for $\eta^{(2)}$. Suppose that the first-order solution we are interested in is $\eta^{(1)} = ae^{-i\omega_0 t}$, already introducing the complex arithmetic that will be used in the sequel also. This solution is found for the initial values $\eta^{(1)}(0) = a$, $\partial_t \eta^{(1)}(0) = -i\omega_0 a$. Then the equation for $\eta^{(2)}$, i.e. $\mathcal{L}\eta^{(2)} = a^2 e^{-2i\omega_0 t}$ has as particular solution: $\eta_p^{(2)} = Ae^{-2i\omega_0 t}$ with $A = -a^2/(3\omega_0^2)$. This particular solution is the equivalent of a ‘bound wave’ mentioned above: it comes inevitably with the first-order solution $\eta^{(1)}$. However, $\eta_p^{(2)}$ will change the initial condition; forcing it to remain unchanged could be done by adding a solution $\eta_h^{(2)}$ of the homogeneous equation: $\mathcal{L}\eta_h^{(2)} = 0$ that cancels the particular solution at $t = 0$, explicitly: $\eta_h^{(2)} = -(\frac{3}{2}Ae^{-i\omega_0 t} - \frac{1}{2}Ae^{i\omega_0 t})$. This homogeneous solution corresponds to the second-order free wave mentioned above. To avoid this solution to be present, the initial value has to be taken like:

$$\eta(0) = \epsilon a + \epsilon^2 A; \quad \partial_t \eta(0) = -i\epsilon\omega_0 a - 2i\epsilon^2\omega_0 A.$$

The second-order terms in ϵ in these initial conditions are similar to the second-order steering of the flap motion for the signalling problem.*

Besides the two difficult aspects of nature, dispersion and nonlinearity, the precise description of the signal is also quite involved, since the signal has to be described on a moving boundary, the flap, which complicates matters also. For the rest of this paper, we will describe the major details of this procedure. The next section presents the BVP for the wave generation problem. Section 3 and Section 4 discuss the first- and the second-order wave generation theory, respectively. The final section gives some conclusions about this paper. The results can also be found in Dean and Dalrymple [1] for the first-order theory as well as in Schäffer [2] for the second-order theory, but our presentation is less technical and emphasises the major steps.

2. GOVERNING EQUATION

Let $\mathbf{u} = (u, w) = (\partial_x \phi, \partial_z \phi)$ define the velocity potential function $\phi = \phi(x, z, t)$ in a Cartesian coordinate system (x, z) . Let also $\eta = \eta(x, t)$, $\Xi = \Xi(z, t) = f(z)S(t)$, g , h , and t denote surface wave elevation, wavemaker position, gravitational acceleration, still water depth and time, respectively. The governing equation for the velocity potential is the Laplace equation

$$\partial_x^2 \phi + \partial_z^2 \phi = 0, \quad \text{for } x \geq \Xi(z, t), \quad -h \leq z \leq \eta(x, t);$$

that results from the assumption that water (in a good approximation) is incompressible: $\nabla \cdot \mathbf{u} = 0$. The dynamic and kinematic free surface boundary conditions (DFSBC and KFSBC), the kinematic boundary condition at the wavemaker (KWMBC), and the bottom boundary condition (BBC) are given by

DFSBC :	$\partial_t \phi + \frac{1}{2} \nabla \phi ^2 + g\eta = 0$	at $z = \eta(x, t)$;
KFSBC :	$\partial_t \eta + \partial_x \eta \partial_x \phi - \partial_z \phi = 0$	at $z = \eta(x, t)$;
KWMBC :	$\partial_x \phi - f(z)S'(t) - f'(z)S(t)\partial_z \phi = 0$	at $x = \Xi(z, t)$;
BBC :	$\partial_z \phi = 0$	at $z = -h$.

*Just as in this example, the hierarchy of equations also continues for the BVP: there will also be the third and higher order contributions, and bound and free waves in each order. A higher-order steering than the second-order one has not been done until now, since the effects are smaller, although there are some exceptions.

The DFSBC is obtained from Bernoulli's equation, the KFSBC and the KWMBBC are derived by applying the material derivative to the surface elevation and wavemaker motion, respectively. The BBC is obtained from the fact that water neither comes in nor goes out of the wave tank. Note that the DFSBC and KFSBC are nonlinear boundary conditions prescribed at a yet unknown and moving free surface $z = \eta(x, t)$. The elevation, potential and wavemaker position are given by the following series expansions

$$\begin{aligned}\eta &= \epsilon\eta^{(1)} + \epsilon^2\eta^{(2)} + \epsilon^3\eta^{(3)} + \dots \\ \phi &= \epsilon\phi^{(1)} + \epsilon^2\phi^{(2)} + \epsilon^3\phi^{(3)} + \dots \\ S &= \epsilon S^{(1)} + \epsilon^2 S^{(2)} + \epsilon^3 S^{(3)} + \dots,\end{aligned}$$

where ϵ is a small parameter, a measure of the surface elevation nonlinearity.

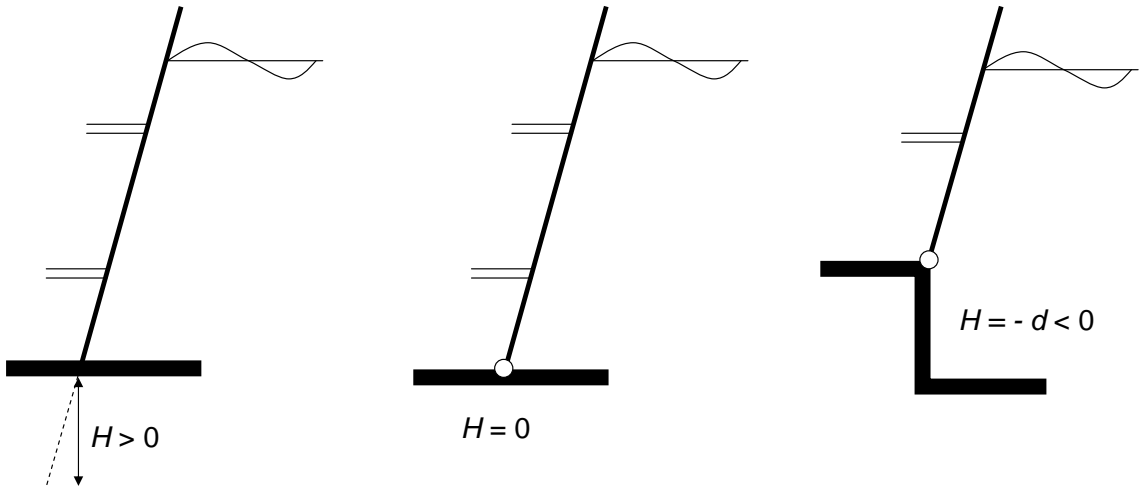


Figure 1: The flap type of wavemaker with different center of rotations: below the bottom (left), at the bottom (middle), and above the bottom (right).

The wavemaker we will consider is a rotating flap, see Figure 1. It is given by $\Xi(z, t) = f(z)S(t)$, where $f(z)$ describes the geometry of the wavemaker:

$$f(z) = \begin{cases} 1 + \frac{z}{h+H}, & \text{for } -(h-d) \leq z \leq 0; \\ 0, & \text{for } -h \leq z < -(h-d). \end{cases} \quad (1)$$

Note that $f(z)$ is given by design, and $S(t)$ is the wavemaker motion that can be controlled externally to generate different types of waves. The center of rotation is at $z = -(h+H)$. If the center of rotation is at or below the bottom, then $d = 0$ and in fact, we do not have the second case of (1). If the center of rotation is at a height d above the bottom, then $d = -H$.

3. FIRST-ORDER WAVE GENERATION THEORY

In this section, we solve a homogeneous BVP for the first-order wave generation theory. By prescribing the first-order wavemaker motion as a linear superposition of monochromatic frequencies,

we find the generated surface elevation also as a linear superposition of monochromatic modes. After applying the Taylor series expansion of the potential function ϕ around $x = 0$ and $z = 0$ as well as applying the series expansion method, the first-order potential function has to satisfy the Laplace equation

$$\partial_x^2 \phi^{(1)} + \partial_z^2 \phi^{(1)} = 0, \quad \text{for } x \geq 0, \quad -h \leq z \leq 0. \quad (2)$$

We also obtain the BVP for the first-order wave generation theory at the lowest expansion order. It reads

$$\begin{aligned} g\eta^{(1)} + \partial_t \phi^{(1)} &= 0, & \text{at } z = 0; \\ \partial_t \eta^{(1)} - \partial_z \phi^{(1)} &= 0, & \text{at } z = 0; \\ \partial_x \phi^{(1)} - f(z) \frac{dS^{(1)}}{dt} &= 0, & \text{at } x = 0; \\ \partial_z \phi^{(1)} &= 0, & \text{at } z = -h. \end{aligned} \quad (3)$$

By combining the DFSBC and the DFSBC at $z = 0$ (3), we obtain the first-order homogeneous free surface boundary condition

$$g\partial_z \phi^{(1)} + \partial_t^2 \phi^{(1)} = 0, \quad \text{at } z = 0. \quad (4)$$

We look for the so-called monochromatic waves

$$\phi^{(1)}(x, z, t) = \psi(z)e^{-i\theta(x, t)},$$

where $\theta(x, t) = kx - \omega t$. Then from the Laplace equation (2), we have $\psi''(z) - k\psi(z) = 0$, for $-h \leq z \leq 0$. Applying the BBC leads to $\psi(z) = \alpha \cosh k(z + h)$, $\alpha \in \mathbb{C}$. From the combined free surface condition (4), we obtain a relation between the wavenumber k and frequency ω , known as the *linear dispersion relation* (LDR), explicitly given by

$$\omega^2 = gk \tanh kh. \quad (5)$$

Let us assume that the first-order wavemaker motion $S^{(1)}(t)$ is given by a harmonic function with frequency ω_n and maximum stroke $|S_n|$ from an equilibrium position, represented in complex notation as

$$S^{(1)}(t) = \sum_{n=1}^{\infty} -\frac{1}{2}iS_n e^{i\omega_n t} + \text{c.c.},$$

where c.c. denotes the complex conjugate of the preceding term. Since this ‘first-order steering’ contains an infinite number of discrete frequencies ω_n , it motivates us to write a general solution for the potential function by linear superposition of discrete spectrum. By choosing the arbitrary spectral coefficient $\alpha = \frac{ig}{2\omega_n} \frac{C_n}{\cosh k_n h}$, the first potential function is found to be

$$\phi^{(1)}(x, z, t) = \sum_{n=1}^{\infty} \frac{ig}{2\omega_n} C_n \frac{\cosh k_n(z + h)}{\cosh k_n h} e^{-i\theta_n(x, t)} + \text{c.c.},$$

where $\theta_n(x, t) = k_n x - \omega_n t$, with wavenumber-frequency pairs (k_n, ω_n) , $n \in \mathbb{Z}$ satisfying the LDR (5). For a continuous spectrum, the summation is replaced by an integral. Allowing the wavenumber to be complex valued, the LDR becomes

$$\omega_n^2 = gk_{nj} \tanh k_{nj} h, \quad j \in \mathbb{N}_0.$$

For $j = 0$, the wavenumber is real and it corresponds to the propagating mode of the surface wave elevation. For $j \in \mathbb{N}$, the wavenumbers are purely imaginary, and thus $ik_{nj} \in \mathbb{R}$. Since we are interested in the decaying solution, we choose $ik_{nj} > 0$ and hence the modes of these wavenumbers are called the *evanescent modes*. As a consequence, the first-order potential function can now be written as

$$\phi^{(1)}(x, z, t) = \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \frac{ig}{2\omega_n} C_{nj} \frac{\cosh k_{nj}(z+h)}{\cosh k_{nj}h} e^{-i\theta_{nj}(x,t)} + \text{c.c.}, \quad (6)$$

where $\theta_{nj}(x, t) = k_{nj}x - \omega_n t$.

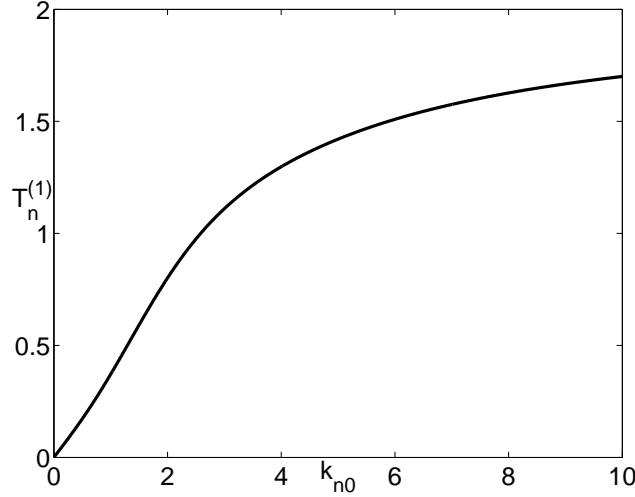


Figure 2: The first-order transfer function plot as a function of wavenumber k_{n0} , for the case that the water depth is $h = 1$ and the center of rotation is at $d = \frac{1}{3}h$ above the tank floor.

Furthermore, applying the KWMB (3), integrating along the water depth, and using the property that $\left\{ \cosh k_{nj}(z+h), \cosh k_{nl}(z+h), j, l \in \mathbb{N}_0 \right\}$ is a set of orthogonal functions for $j \neq l$, we can find the surface wave complex-valued amplitude C_{nj} as follows

$$\begin{aligned} C_{nj} &= \frac{\omega_n^2 S_n}{g k_{nj}} \cosh k_{nj} h \frac{\int_{-h}^0 f(z) \cosh k_{nj}(z+h) dz}{\int_{-h}^0 \cosh^2 k_{nj}(z+h) dz} \\ &= \frac{4S_n \sinh k_{nj} h}{k_{nj}(h+H)} \frac{k_{nj}(h+H) \sinh k_{nj} h + \cosh k_{nj} d - \cosh k_{nj} h}{2k_{nj} h + \sinh(2k_{nj} h)}, \quad j \in \mathbb{N}_0. \end{aligned}$$

Finally, the first-order surface elevation can be found from the DFSBC (3), and is given as follows:

$$\eta^{(1)}(x, t) = \sum_{n=-\infty}^{\infty} \sum_{j=0}^{\infty} \frac{1}{2} C_{nj} e^{-i\theta_{nj}(x,t)} + \text{c.c.}$$

This first-order theory can also be found in Dean and Dalrymple [1].

REMARK 1.

For ‘practical’ purposes, it is useful to introduce the so-called *transfer function* or *frequency response* of a system. It is defined as the ratio of the output and the input of a system. In our wave generation problem, we have a system with a wavemaker motion as input and the surface wave amplitude as output. Therefore, the first-order transfer function $T_n^{(1)}$ is defined as the ratio between the surface wave amplitude of the propagating mode C_{n0} as output and the maximum stroke $|S_n|$ as input, explicitly given by

$$T_n^{(1)} = 4 \frac{\sinh k_{n0}h}{k_{n0}(h+H)} \frac{k_{n0}(h+H) \sinh k_{n0}h + \cosh k_{n0}d - \cosh k_{n0}h}{2k_{n0}h + \sinh 2k_{n0}h}.$$

Figure 2 shows the first-order transfer function plot as function of wavenumber k_{n0} for a given water depth h and the center of rotation d . For increasing k_{n0} , which also means increasing frequency ω_n , the transfer function is monotonically increasing as well. It increases faster for smaller values of k_{n0} and slower for larger values of k_{n0} , approaching the asymptotic limit of $T_n^{(1)} = 2$ for $k_{n0}h \rightarrow \infty$.

4. SECOND-ORDER WAVE GENERATION THEORY

In this section, we solve a nonhomogeneous BVP for the second-order wave generation theory. Due to the nonhomogeneous boundary condition at the free surface, which causes interactions between each possible pair of first-order wave components, the resulting surface wave elevation has a second-order effect, known as the bound wave component. Furthermore, due to the first-order wavemaker motion and the boundary condition at the wavemaker, the generated wave also has another second-order effect, namely the free wave component. The latter component is undesired since it results in a spatially inhomogeneous wave field due to the different propagation velocities of bound wave and free wave components with the same frequency. Therefore, in order to prevent the free wave component to be generated, we add an additional second-order bound wave effect to the flap motion. This process is known as ‘second-order steering’ of the wavemaker motion. More details about this theory, including an experimental verification, can be found in Schäffer [2]. For the history of wave generation theory, see also references in this paper.

Taking terms of the second-order in the series expansion, we obtain the BVP for the second-order wave generation theory. The second-order potential function also satisfies the Laplace equation

$$\partial_x^2 \phi^{(2)} + \partial_z^2 \phi^{(2)} = 0, \quad \text{for } x \geq 0, \quad -h \leq z \leq 0.$$

Almost all the second-order boundary conditions now become nonhomogeneous

$$\begin{aligned} g\eta^{(2)} + \partial_t \phi^{(2)} &= -\left(\eta^{(1)} \partial_{tz}^2 \phi^{(1)} + \frac{1}{2} |\nabla \phi^{(1)}|^2\right), & \text{at } z = 0; \\ \partial_t \eta^{(2)} - \partial_z \phi^{(2)} &= \eta^{(1)} \partial_z^2 \phi^{(1)} - \partial_x \eta^{(1)} \partial_x \phi^{(1)}, & \text{at } z = 0; \\ \partial_x \phi^{(2)} - f(z) \frac{dS^{(2)}}{dt} &= S^{(1)}(t) \left(f'(z) \partial_z \phi^{(1)} - f(z) \partial_x^2 \phi^{(1)}\right), & \text{at } x = 0; \\ \partial_z \phi^{(2)} &= 0, & \text{at } z = -h. \end{aligned} \tag{7}$$

By combining the DFSBC and KFSBC of (7) at $z = 0$, we have the second-order nonhomogeneous free surface boundary condition

$$g\partial_z \phi^{(2)} + \partial_t^2 \phi^{(2)} = \text{RHS}_1, \quad \text{at } z = 0. \tag{8}$$

Using the first-order potential function (6), RHS_1 is explicitly given by

$$\begin{aligned} \text{RHS}_1 &= - \left(\frac{\partial}{\partial t} |\nabla \phi^{(1)}|^2 + \eta^{(1)} \frac{\partial}{\partial z} \left[g \frac{\partial \phi^{(1)}}{\partial z} + \frac{\partial^2 \phi^{(1)}}{\partial t^2} \right] \right) \Big|_{z=0} \\ &= \sum_{m,n=1}^{\infty} \sum_{l,j=0}^{\infty} \left(A_{mnlj}^+ e^{-i(\theta_{ml} + \theta_{nj})} + A_{mnlj}^- e^{-i(\theta_{ml} - \theta_{nj}^*)} \right) + \text{c.c.}, \end{aligned}$$

where

$$\begin{aligned} \frac{A_{mnlj}^+}{C_{ml}C_{nj}} &= \frac{1}{4i} \left[(\omega_m + \omega_n) \left(g^2 \frac{k_{ml}k_{nj}}{\omega_m\omega_n} - \omega_m\omega_n \right) + \frac{g^2}{2} \left(\frac{k_{ml}^2}{\omega_m} + \frac{k_{nj}^2}{\omega_n} \right) - \frac{1}{2}(\omega_m^3 + \omega_n^3) \right], \\ \frac{A_{mnlj}^-}{C_{ml}C_{nj}^*} &= \frac{1}{4i} \left[(\omega_m - \omega_n) \left(g^2 \frac{k_{ml}k_{nj}^*}{\omega_m\omega_n} + \omega_m\omega_n \right) + \frac{g^2}{2} \left(\frac{k_{ml}^2}{\omega_m} - \frac{k_{nj}^2}{\omega_n} \right) - \frac{1}{2}(\omega_m^3 - \omega_n^3) \right]. \end{aligned}$$

In order to find the bound wave component, the free wave component, and to apply the second-order steering wavemaker motion, we split the second-order BVP (7) into three BVPs. For that purpose, the second-order potential function is split into three components as follows

$$\phi^{(2)}(x, z, t) = \phi^{(21)}(x, z, t) + \phi^{(22)}(x, z, t) + \phi^{(23)}(x, z, t).$$

Now the corresponding BVP for the first component of the potential function $\phi^{(21)}$ reads

$$\begin{aligned} g\partial_z\phi^{(21)} + \partial_t^2\phi^{(21)} &= \text{RHS}_1, & \text{at } z = 0; \\ \partial_z\phi^{(21)} &= 0, & \text{at } z = -h. \end{aligned} \quad (9)$$

The corresponding BVP for the second component of the potential function $\phi^{(22)}$ reads

$$\begin{aligned} g\partial_z\phi^{(22)} + \partial_t^2\phi^{(22)} &= 0, & \text{at } z = 0; \\ \partial_x\phi^{(22)} &= S^{(1)}(t) \left(f'(z)\partial_z\phi^{(1)} - f(z)\partial_x^2\phi^{(1)} \right) - \partial_x\phi^{(21)}, & \text{at } x = 0; \\ \partial_z\phi^{(22)} &= 0, & \text{at } z = -h. \end{aligned} \quad (10)$$

And the BVP for the third component of the potential function $\phi^{(23)}$ reads

$$\begin{aligned} g\partial_z\phi^{(23)} + \partial_t^2\phi^{(23)} &= 0, & \text{at } z = 0; \\ \partial_x\phi^{(23)} &= f(z)\frac{dS^{(2)}}{dt}, & \text{at } x = 0; \\ \partial_z\phi^{(23)} &= 0, & \text{at } z = -h. \end{aligned} \quad (11)$$

By taking the Ansatz for the first part of the second-order potential function $\phi^{(21)}$ as follows

$$\begin{aligned} \phi^{(21)}(x, z, t) &= \sum_{m,n=1}^{\infty} \sum_{l,j=0}^{\infty} B_{mnlj}^+ \frac{\cosh(k_{ml} + k_{nj})(z + h)}{\cosh(k_{ml} + k_{nj})h} e^{-i(\theta_{ml} + \theta_{nj})} \\ &+ B_{mnlj}^- \frac{\cosh(k_{ml} - k_{nj}^*)(z + h)}{\cosh(k_{ml} - k_{nj}^*)h} e^{-i(\theta_{ml} - \theta_{nj}^*)} + \text{c.c.}, \end{aligned}$$

then we can derive the corresponding coefficients to be:

$$\begin{aligned} B_{mnlj}^+ &= \frac{A_{mnlj}^+}{\Omega^2(k_{ml} + k_{nj}) - (\omega_m + \omega_n)^2}, \\ B_{mnlj}^- &= \frac{A_{mnlj}^-}{\Omega^2(k_{ml} - k_{nj}^*) - (\omega_m - \omega_n)^2}. \end{aligned}$$

This first component of the second-order potential function will contribute the bound wave component to the second-order surface wave elevation $\eta^{(2)}$. For $j = 0$, the wave component is a propagating mode and for $j \in \mathbb{N}$, it consists of evanescent modes. Since the wavenumbers $k_{mj} + k_{nj}$ and $k_{mj} + k_{nj}^*$, $j \in \mathbb{N}_0$ do not satisfy the LDR with frequencies $\omega_m \pm \omega_n$, then the denominator part of B_{mnlj}^\pm will never vanish and thus the potential function is a bounded function.

Let the right-hand side of the boundary condition at the wavemaker for the second BVP (10) be denoted by RHS_2 , which is expressed as

$$\text{RHS}_2 = \sum_{m,n=1}^{\infty} \sum_{l,j=0}^{\infty} \left(F_{mnlj}^+(z) e^{i(\omega_m + \omega_n)t} + F_{mnlj}^-(z) e^{i(\omega_m - \omega_n)t} \right) + \text{c.c.},$$

where

$$\begin{aligned} F_{mnlj}^+(z) &= \frac{g}{8\omega_n} \frac{S_m k_{nj} C_{nj}}{\cosh k_{nj} h} [f'(z) \sinh k_{nj}(z+h) + k_{nj} f(z) \cosh k_{nj}(z+h)] \\ &\quad + i(k_{ml} + k_{nj}) B_{mnlj}^+ \frac{\cosh(k_{ml} + k_{nj})(z+h)}{\cosh(k_{ml} + k_{nj})h}, \\ F_{mnlj}^-(z) &= -\frac{g}{8\omega_n} \frac{S_m k_{nj}^* C_{nj}^*}{\cosh k_{nj}^* h} [f'(z) \sinh k_{nj}^*(z+h) + k_{nj}^* f(z) \cosh k_{nj}^*(z+h)] \\ &\quad + i(k_{ml} - k_{nj}^*) B_{mnlj}^- \frac{\cosh(k_{ml} - k_{nj}^*)(z+h)}{\cosh(k_{ml} - k_{nj}^*)h}. \end{aligned}$$

Let the Ansatz for the second component of the second-order potential function $\phi^{(22)}$ be

$$\begin{aligned} \phi^{(22)}(x, z, t) &= \sum_{m,n=1}^{\infty} \sum_{l,j=0}^{\infty} \left(\frac{i g P_{mnlj}^+}{2(\omega_m + \omega_n)} \frac{\cosh K_{mnlj}^+(z+h)}{\cosh K_{mnlj}^+ h} e^{-i(K_{mnlj}^+ x - (\omega_m + \omega_n)t)} \right. \\ &\quad \left. + \frac{i g P_{mnlj}^-}{2(\omega_m - \omega_n)} \frac{\cosh K_{mnlj}^-(z+h)}{\cosh K_{mnlj}^- h} e^{-i(K_{mnlj}^- x - (\omega_m - \omega_n)t)} \right) + \text{c.c.}, \end{aligned}$$

where the wavenumbers K_{mnlj}^\pm , $j \in \mathbb{N}_0$ and frequencies $\omega_m \pm \omega_n$ satisfy the LDR. Using the property that $\left\{ \cosh K_{mnlj}^\pm(z+h), \cosh K_{mnlj'}^\pm(z+h), l, l', j, j' \in \mathbb{N}_0 \right\}$ is a set of orthogonal functions for $l \neq l'$ and $j \neq j'$, we find the coefficients P_{mnlj}^\pm as follow

$$\begin{aligned} P_{mnlj}^\pm &= \frac{2(\omega_m \pm \omega_n) \cosh K_{mnlj}^\pm h \int_{-h}^0 F_{mnlj}^\pm(z) \cosh K_{mnlj}^\pm(z+h) dz}{g K_{mnlj}^\pm \int_{-h}^0 \cosh^2 K_{mnlj}^\pm(z+h) dz} \\ &= 8 \frac{K_{mnlj}^\pm \sinh K_{mnlj}^\pm h \int_{-h}^0 F_{mnlj}^\pm(z) \cosh K_{mnlj}^\pm(z+h) dz}{\omega_m \pm \omega_n 2K_{mnlj}^\pm h + \sinh(2K_{mnlj}^\pm h)}. \end{aligned} \tag{12}$$

The second component of the second-order potential function $\phi^{(22)}$ will give contributions to the free wave component of the second-order surface wave elevation $\eta^{(2)}$. This component arises due to the boundary condition at the wavemaker caused by the first-order wavemaker motion. Since the desired surface elevation is only the bound wave component, we want to get rid this term, especially the propagating mode. The evanescent modes vanish anyway after they evolve far away from the wavemaker. By prescribing the second-order wavemaker motion such that the propagating mode of the third component $\phi^{(23)}$ will cancel the same mode of the second one $\phi^{(22)}$, then far from the wavemaker we have the desired bound wave component only.

Let the second-order wavemaker motion be given by

$$S^{(2)}(t) = \sum_{m,n=1}^{\infty} -\frac{1}{2}i \left(S_{mn}^{+} e^{i(\omega_m + \omega_n)t} + S_{mn}^{-} e^{i(\omega_m - \omega_n)t} \right) + \text{c.c.}$$

Let also the Ansatz for the third component of the second-order potential function $\phi^{(23)}$ be

$$\begin{aligned} \phi^{(23)}(x, z, t) = & \sum_{m,n=1}^{\infty} \sum_{l,j=0}^{\infty} \left(\frac{i g Q_{mnlj}^{+}}{2(\omega_m + \omega_n)} \frac{\cosh K_{mnlj}^{+}(z+h)}{\cosh K_{mnlj}^{+}h} e^{-i(K_{mnlj}^{+}x - (\omega_m + \omega_n)t)} \right. \\ & \left. + \frac{i g Q_{mnlj}^{-}}{2(\omega_m - \omega_n)} \frac{\cosh K_{mnlj}^{-}(z+h)}{\cosh K_{mnlj}^{-}h} e^{-i(K_{mnlj}^{-}x - (\omega_m - \omega_n)t)} \right) + \text{c.c.}, \end{aligned}$$

where $\Omega(K_{mnlj}^{\pm}) = \omega_m \pm \omega_n$. Using the orthogonality property again, we find the coefficients Q_{mnlj}^{\pm} as follows

$$\begin{aligned} Q_{mnlj}^{\pm} &= S_{mn}^{\pm} \sinh K_{mnlj}^{\pm} h \frac{\int_{-h}^0 f(z) \cosh K_{mnlj}^{\pm}(z+h) dz}{\int_{-h}^0 \cosh^2 K_{mnlj}^{\pm}(z+h) dz} \\ &= \frac{4S_{mn}^{\pm} \sinh K_{mnlj}^{\pm} h}{K_{mnlj}^{\pm}(h+H)} \frac{K_{mnlj}^{\pm}(h+H) \sinh K_{mnlj}^{\pm} h + \cosh K_{mnlj}^{\pm} d - \cosh K_{mnlj}^{\pm} h}{2K_{mnlj}^{\pm} h + \sinh(2K_{mnlj}^{\pm} h)}. \end{aligned} \quad (13)$$

To have the propagating mode of the free wave from the second ($\phi^{(22)}$) and the third ($\phi^{(23)}$) components cancel each other, we must require $P_{mn00} + Q_{mn00} = 0$, which leads to the following second-order wavemaker motion, known as ‘second-order steering’:

$$S_{mn}^{\pm} = \frac{2(K_{mn00}^{\pm})^2 I_{mn00}^{\pm}(h+H)}{(\omega_m \pm \omega_n) (\cosh K_{mn00}^{\pm} h - \cosh K_{mn00}^{\pm} d - K_{mn00}^{\pm}(h+l) \sinh K_{mn00}^{\pm} h)},$$

where

$$I_{mn00}^{\pm} = \int_{-h}^0 F_{mn00}^{\pm}(z) \cosh K_{mn00}^{\pm}(z+h) dz.$$

Therefore, with this choice of the second-order wavemaker motion, the second-order potential function can be written as

$$\phi^{(2)}(x, z, t) = \phi_{\text{propagating}}^{(2)}(x, z, t) + \phi_{\text{evanescent}}^{(2)}(x, z, t),$$

where

$$\begin{aligned}\phi_{\text{propagating}}^{(2)} &= \phi_{\text{bound wave, propagating}}^{(21)} \\ \phi_{\text{evanescent}}^{(2)} &= (\phi_{\text{bound wave}}^{(21)} + \phi_{\text{free wave}}^{(22)} + \phi_{\text{free wave}}^{(23)})_{\text{evanescent}}.\end{aligned}$$

Consequently, from the second-order DFSBC (7), we find the second-order surface wave elevation. It can be written as follows

$$\eta^{(2)}(x, t) = \eta_{\text{propagating}}^{(2)} + \phi_{\text{evanescent}}^{(2)},$$

where

$$\eta_{\text{propagating}}^{(2)} = \sum_{m,n=1}^{\infty} D_{mn00}^+ e^{-i(\theta_{m0} + \theta_{n0})} + D_{mn00}^- e^{-i(\theta_{m0} - \theta_{n0})},$$

and

$$\begin{aligned}\eta_{\text{evanescent}}^{(2)} &= \sum_{m,n=1}^{\infty} \sum_{\substack{j=1, \\ l=0}}^{\infty} \left(D_{mnlj}^+ e^{-i(\theta_{ml} + \theta_{nj})} + D_{mnlj}^- e^{-i(\theta_{ml} - \theta_{nj}^*)} \right) \\ &+ \frac{1}{2} (P_{mnlj}^+ + Q_{mnlj}^+) e^{-i(K_{mnlj}^+ x - (\omega_m + \omega_n)t)} \\ &+ \frac{1}{2} (P_{mnlj}^- + Q_{mnlj}^-) e^{-i(K_{mnlj}^- x - (\omega_m - \omega_n)t)} + \text{c.c.},\end{aligned}$$

where for $l, j \in \mathbb{N}_0$:

$$\begin{aligned}D_{mnlj}^+ &= -\frac{1}{g} \left[i(\omega_m + \omega_n) B_{mnlj}^+ + \frac{1}{8} \left(g^2 \frac{k_{ml} k_{nj}}{\omega_m \omega_n} - \omega_m \omega_n - (\omega_m^2 + \omega_n^2) \right) C_{ml} C_{nj} \right], \\ D_{mnlj}^- &= -\frac{1}{g} \left[i(\omega_m - \omega_n) B_{mnlj}^- + \frac{1}{8} \left(g^2 \frac{k_{ml} k_{nj}^*}{\omega_m \omega_n} + \omega_m \omega_n - (\omega_m^2 + \omega_n^2) \right) C_{ml} C_{nj}^* \right].\end{aligned}$$

We have seen that the first-order surface wave elevation consists of a linear superposition of monochromatic frequencies. However, due to nonlinear effects, nonhomogeneous BVP, and interactions of the first-order wave components, the second-order surface elevation is composed by a superposition of bichromatic frequencies $\omega_m \pm \omega_n$. The components with frequency $\omega_m + \omega_n$ are called the ‘superharmonics’ and those with frequency $|\omega_m - \omega_n|$ are called the ‘subharmonics’.

REMARK 2.

Similar to the first-order wave generation theory, we can define a second-order transfer function as well. A detailed formula for this transfer function can be found in Schäffer [2].

5. CONCLUSIONS

We have discussed the theory for wave generation based on the fully nonlinear water wave equation. We solved a nonlinear BVP by the series expansion method. Using this method, the problem turns into a set of linear BVPs at each expansion order. The lowest order gives a homogeneous BVP and the higher orders give nonhomogeneous ones with known, depending on previous solutions, right-hand sides. In this paper, we focussed on the wave generation theory up to the second-order.

Based on the first-order wave generation theory, we describe the surface wave fields as the superposition of monochromatic waves. Due to the BVP, the wavenumbers and frequencies of this wave field are related by the LDR. By prescribing the first-order steering of the wavemaker as a linear superposition of harmonic motions, we found that the first-order surface elevation is simply a linear superposition of the corresponding monochromatic waves. Furthermore, the wavemaker transfer function is also introduced for practical purposes in the laboratory.

For the second-order wave generation theory, we have solved a nonhomogeneous BVP. Due to the interactions between each pair of first-order wave components, the second-order wave field has bound wave components. Additionally, due to the first-order steering of the wavemaker motion and the boundary condition at the wavemaker, a free wave component is also generated, which is undesired. Therefore, we prevent it by controlling this second-order wavemaker motion. By applying this second-order steering, the resulting surface wave field contains only the desired bound wave components. Similarly, one can find the second-order transfer function for the relationship between the first-order and the second-order motions.

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